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BOUNDARY OF MONOTONIC AND OSCILLATORY CONVECTIVE STABILITY OF A HORIZONTAL FLUID LAYER

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The problem of small oscillations of a heat-conducting fluid which occupies a horizontal layer and is close to mechanical equilibrium is examined here. It is assumed that the layer is heated from above, so that the fluid is stably stratified. As is known [1, 2], for sufficiently high viscosity, all modes are monotonically damped (the decrements are positive), but if the viscosity is low enough, then there are also oscillatory modes, which correspond to complex decrements with positive real and nonzero imaginary parts.

Here the limiting case of infinitely large Prandtl σ and Rayleigh R numbers is studied, the Grashoff number $G = R/\sigma$ being finite and fixed in value. The problem reduces to analysis of the spectral boundary-value problem for a fourth-order ordinary differential equation which is nonlinear in the spectral parameter, the decrement λ. The problem contains as auxiliary parameters the wavenumber α and G. For fixed α and G, it is easily established that there exists a countable set $\{\lambda_n\}_{n=1}^{\infty}$ of eigenvalues. In this case, the eigenvalues are all real if G is sufficiently small. When G, as it grows, reaches a definite critical value, there appear a series of pairs of complex-conjugate eigennumbers λ which, as usual, are determined from the appropriate transcendental equation. To analyze the equation, the method of one-dimensional perturbations (perturbations of boundary conditions) is applied. This method was used by Jeffries [3] in the convection problem. The method leads directly to an expansion of the left side of the transcendental equation in partial fractions, which facilitates study: specificaly, it helps in isolating the roots.

The minimum values in α of the critical Grashoff numbers G_n for the appropriate values of α and λ are determined. These are found separately for the even and odd modes with respect to the transverse variable. The asymptotes to G_n for $n \to \infty$ are constructed. It is remarkable that even for n = 1, the asymptotics yield good accuracy.

There are grounds for believing that the critical value of the Grashoff number $G_* = 729$, which results in the first appearance of an oscillatory mode, corresponds to the transition of turbulent convection at infinitely large Prandtl numbers [4].

1. Problem Statement. The stability spectrum ("spectrum of small oscillations") is determined in this case by the boundary-value problem

$$(D^2-\alpha^2)^2\varphi+\alpha^2R\theta=-\lambda(D^2-\alpha^2)\varphi; \quad (1.1)$$

$$(D^2-\alpha^2)\theta+\varphi=-\lambda\sigma\theta; \quad (1.2)$$

$$\varphi=\varphi'=\theta=0\ (z=\mp 1). \quad (1.3)$$

Here R is the Rayleigh number with a minus sign, so that positive R corresponds to stability; α^2 is the square of the modulus of the horizontal wave vector; $D = d/dz$; λ is the com-

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plex spectral parameter (the sign chosen so that stability corresponds to Re $\lambda > 0$); ϕ, θ are the complex amplitudes of the normal oscillations of, respectively, the vertical component of the velocity and the temperature.

It is well known that for R, $\sigma > 0$, the spectrum of the boundary-value problem (1.1)-(1.3) lies in the right half plane. For sufficiently small R, it is real, but can become complex when R increases to attain a definite value which depends on σ. When this occurs, the real eigenvalues λ merge, and with further growth in R are transformed into complex-conjugate pairs. We are interested in the case where $R \to \infty$, $\sigma \to \infty$ in such a way that $G = R/\sigma$ remains finite and fixed in value.

We make the substitution $\phi \to \sigma\phi$ in (1.1)-(1.3). For $\sigma \to \infty$, we obtain from (1.2) $\theta = -\phi/\lambda$. Substitution into (1.1), (1.3) gives the boundary-value problem

$$L^2\varphi - (\alpha^2 G/\lambda)\varphi = -\lambda L\varphi,\ L = D^2 - \alpha^2; \tag{1.4}$$

$$\varphi = \varphi' = 0\ (z = \mp 1). \tag{1.5}$$

Because of the invariance of (1.4), (1.5) with respect to mirror symmetry $z \to -z$, the eigenfunctions and adjoint functions are divided into even and odd functions. It is for this reason that we consider the symmetry of the interval with respect to z. It must be kept in mind that the critical values of G found here must be multiplied by 16 to obtain values for a layer of unit thickness.

We can find the transit values of G for which oscillatory modes appear (complex-conjugate λ pairs) by the criterion of two-fold or greater multiplicity of the eigenvalues λ. A mere single multiplicity is, however, not enough. If it is brought about by the existence of a pair of independent eigenfunctions, then in general, as easily seen from perturbation theory, both merging eigennumbers are either real along both sides of such values of G, or are complex. The generation of complex-conjugate pairs occurs if for the given value of G there is a Jordan cell: there appears an adjoint function [5, 6].

This difficulty in analyzing (1.4), (1.5) is easily circumvented if we consider the even and odd eigenfunctions separately: it is subsequently shows that for both the class of even and odd functions, the corresponding characteristic subspace is always one-dimensional, and multiplicity can occur only with the appearance of an adjoint function.

2. The Transcendental Equation. We obtain from (1.4) the characteristic equation

$$(k^2 - \alpha^2)^2 + \lambda(k^2 - \alpha^2) - \alpha^2 G/\lambda = 0. \tag{2.1}$$

If the roots are $\mp k_1, \mp k_2$:

$$k_1 = \sqrt{\alpha^2 + l_1},\ k_2 = \sqrt{\alpha^2 + l_2}, \tag{2.2}$$
$$l_{1,2} = (1/2)(-\lambda \pm \sqrt{\lambda^2 + 4\alpha^2 G/\lambda})$$

(ℓ_1 corresponds to the + sign).

For even eigenfunctions, we have $\phi = A \cosh k_1 z + B \cosh k_2 z$, A, B = const. Substituting this in (1.5) and equating the determinant of the resultant system to zero, we obtain

$$k_1 \operatorname{th} k_1 - k_2 \operatorname{th} k_2 = 0, \tag{2.3}$$

since, as is easily verified, $\cosh k_1 \cdot \cosh k_2 \neq 0$.

The odd eigenfunction case is completely analogous: the eigenfunctions are $\phi = C \sinh k_1 z + D \sinh k_2 z$, C, D = const. In place of (2.3), we find

$$k_1 \operatorname{cth} k_1 - k_2 \operatorname{cth} k_2 = 0. \tag{2.4}$$

It is clear that the characteristic subspaces are one-dimensional in both problems.

3. Boundary Conditions Perturbation Method. Change of one of the boundary conditions of the boundary-value problem for the eigenvalues for an ordinary differential equation leads to a one-dimensional perturbation of the operator for this problem or of the appropriate Green operator. In this case, the dispersion equation of the perturbed problem has a very convenient representation. In the theory of self-adjoint boundary-value problems, such an approach is known as the Weinstein method. In [7-9], the method of one-dimensional perturbations is developed in the general, nonself-adjoint case. It is interesting that the special one-dimensional perturbations being considered turn out to be very useful in the

study of the unperturbed problem as well: it is helpful to know the response of the system being studied to external influences.

For the problem of convection in a layer, the method leads to a dispersion relation of the form first applied in [3], where the results were based on a Fourier transform with finite limits. It is also possible to obtain this equation by replacing the hyperbolic tangent and cotangent in (2.3), (2.4) by their expansions in partial fractions. We now derive the dispersion relation by considering (1.4), (1.5) as a perturbation with respect to the boundary-value problem with conditions

$$u = u'' = 0 \ (z = \mp 1), \tag{3.1}$$

corresponding to a non-deformable free boundary. Such a perturbation is one-dimensional in each of the subspaces for the even and odd functions. Let us start with the even modes.

We consider the inhomogeneous boundary-value problem for the equation

$$L^2 u = f \tag{3.2}$$

with conditions (3.1), where f, u are even with respect to z. The function f is represented in the form of a Fourier series

$$f(z) = \sum_{n=1}^{\infty} f_n \cos \delta_n z, \quad \delta_n = (2n-1)\pi/2, \quad f_n = 2\int_0^1 f(z) \cos \delta_n z \, dz. \tag{3.3}$$

We seek the solution u also in the form of a Fourier series, thereby determining the Green's operator $\mathscr{H}$ for problem (3.1), (3.2) ($\gamma_n = \delta_n^2 + \alpha^2$):

$$u(z) = (\mathscr{H} f)(z) = \sum_{n=1}^{\infty} (f_n/\gamma_n^2) \cos \delta_n z. \tag{3.4}$$

Now, from (1.4), (1.5) we infer

$$\varphi = \mathscr{H}(-\lambda L\varphi + (\alpha^2 G/\lambda)\varphi) + \gamma f. \tag{3.5}$$

Here γ is a number (the value of the functional on ϕ)

$$\gamma = \gamma(\varphi) = -[\mathscr{H}(-\lambda L\varphi + (\alpha^2 G/\lambda)\varphi)]'|_{z=1}, \tag{3.6}$$

and f is the even solution of the boundary-value problem

$$L^2 f = 0, \ f(1) = 0, \ f'(1) = 1. \tag{3.7}$$

Solving (3.7), we obtain ($\Delta = (\sinh 2\alpha + 2\alpha)/2$)

$$f(z) = (z \operatorname{sh} \alpha z \operatorname{ch} \alpha - \operatorname{ch} \alpha z \operatorname{sh} \alpha)/\Delta. \tag{3.8}$$

We represent f in the form of Fourier series (3.3) with coefficients

$$f_n = (-1)^n \frac{8\alpha \operatorname{ch}^2 \alpha \delta_n}{(2\alpha + \operatorname{sh} 2\alpha)\gamma_n^2}. \tag{3.9}$$

Equation (3.5) assumes the form

$$\varphi = \gamma[1 - \mathscr{H}(-\lambda L + \alpha^2 G/\lambda)]^{-1} f. \tag{3.10}$$

Substituting (3.10) in (3.6) and using (3.4) and (3.9), we have the dispersion relation for λ:

$$\sum_{n=1}^{\infty} \frac{\delta_n^2}{\gamma_n^2 - \lambda\gamma_n - \alpha^2 G/\lambda} = \sum_{n=1}^{\infty} \frac{\delta_n^2}{\gamma_n^2} - \frac{2\alpha + \operatorname{sh} 2\alpha}{8\alpha \operatorname{ch}^2 \alpha}.$$

Using the well-known relation

$$\frac{\operatorname{th} \alpha}{2\alpha} = \sum_{n=1}^{\infty} \frac{1}{\delta_n^2 + \alpha^2},$$

The dispersion relation takes on the form

$$\sum_{n=1}^{\infty} \frac{\delta_n^2}{\alpha^2 G - H_n} = 0, \quad H_n = \lambda\gamma_n(\gamma_n - \lambda). \tag{3.11}$$

In the above, the standard proofs of convergence and of the validity of termwise series differentiation have been omitted. Note that in using the boundary conditions perturbation method, it is necessary to analyze separately those eigenvalues which do not change when the problem is perturbed (these are called fixed eigenvalues). It will be shown that in this problem, there are no such eigenvalues. We will say that the function f is of nonlocalized distribution, if all of its Fourier coefficients are nonzero in the system of eigenfunctions ϕ_n^0 of the unperturbed problem. Correspondingly, we call $\gamma(\phi)$ a functional of nonlocaized distribution if it is not zero on any of the eigenfunctions ϕ_n^0. From (3.5), it follows that there are fixed eigenvalues if and only if f or γ is not of nonlocalized distribution. But this is impossible, as is easily seen from (3.6) and (3.9), since $\phi_n^0(z) = \cos \delta_n z$.

We turn now to an analysis of (3.11). Let $\lambda > 0$ be fixed in value. It can be directly verified that all $H_n(\lambda)$ which are positive for the given λ, are distinct and grow monotonically with increasing n. Let $n_0(\lambda) = \min\{n\colon\ H_n(\lambda) > 0\}$; for each from the intervals $(H_n\alpha^{-2},\ H_{n+1}\alpha^{-2})$, $n \geq n_0(\lambda)$, the left hand side of (3.11) monotonically dies out as a function of G from $+\infty$ to $-\infty$ (differentiate the expression: the derivative is negative). Thus, (3.11) has a sequence of positive roots $G_n(\alpha, \lambda)$, $n = n_0(\lambda), n_0(\lambda) + 1, \ldots$: $H_n(\lambda) < \alpha^2 G_n(\alpha, \lambda) < H_{n+1}(\lambda)$ (and possibly another positive root on $(0, H_n\alpha^{-2})$).

It is easy to see that for some $\lambda \in (0, \gamma_{n+1})$, the function $G_n(\alpha, \lambda)$ attains a maximum with respect to λ. Computations (the tabulated $G_n(\alpha, \lambda)$) show that such a point is unique. It is clear that this point determines the moment that oscillatory modes appear: if G becomes slightly larger than the maximum value G_n, a pair of real eigenvalues disappears. It is of interest to find the minimum value of such a critical number with respect to α. As a result of this we have the problem: find G_n ($n = 1, 2, \ldots$) satisfying the condition ($\theta = \alpha^2$)

$$G_n = \min_{\theta>0} \max_{\lambda} G_n(\alpha, \lambda), \tag{3.12}$$

and the α_n, λ_n for which a minimax is attained. In this case, G_1 is the value of the first Grashoff number for which there is at least one oscillatory mode.

Everything considered here is done in a completely analogous fashion for the odd modes. The dispersion relation has the same form (3.11), except that $\delta_n = n\pi$. The equalities analogous to (3.12) determine a sequence of values for G which mark the appearance of new (odd) oscillatory modes as G transits these values.

4. Asymptotics of the Critical Grashoff Numbers. Even Modes. We introduce the function $f(k) = k \tanh k$ and rewrite (2.3) in the form

$$f(k_1) = f(k_2). \tag{4.1}$$

The problem consists of finding the quantities (3.12) ($n = 1, 2, \ldots$). The points (θ_n, λ_n) at which a minimax is attained are critical saddle points of the function $G_n(\alpha, \lambda)$. The quantities G_n, θ_n, and λ_n must satisfy the system of equations (4.1) and

$$f'(k_1)\frac{\partial k_1}{\partial \lambda} = f'(k_2)\frac{\partial k_2}{\partial \lambda}, \quad f'(k_1)\frac{\partial k_1}{\partial \theta} = f'(k_2)\frac{\partial k_2}{\partial \theta}. \tag{4.2}$$

Note that $f'(k_1) > 0$, since $k_1 > 0$. Therefore the determinant of (4.2) must vanish

$$\frac{\partial k_1}{\partial \theta}\frac{\partial k_2}{\partial \lambda} - \frac{\partial k_1}{\partial \lambda}\frac{\partial k_2}{\partial \theta} = 0. \tag{4.3}$$

We introduce the parameters m, q, p by setting

$$m = (k^2 - \alpha^2)/\lambda,\ p = \alpha^2/\lambda,\ q = \alpha^2 G/\lambda^3. \tag{4.4}$$

Note that p and q are positive. Because of (2.1), we have for m

$$m^2 + m - q = 0. \tag{4.5}$$

We denote the positive root by m_1, and the negative root by m_2:

$$m_1 = (-1 + \sqrt{1 + 4q})/2,\ m_2 = (1 + \sqrt{1 + 4q})/2. \tag{4.6}$$

It is easy to see that k_2 is purely imaginary: $k_2 = i\hat{k}_2$. In this case

$$k_1 = \sqrt{\lambda(m_1 + p)},\ \hat{k}_2 = \sqrt{\lambda(m_2 - p)}. \tag{4.7}$$

Taking (4.4)-(4.7) into account, Eqs. (4.3) is reduced to

$$1/p + 1/q = 2. \tag{4.8}$$

With the use of (4.4)-(4.8), the first of Eqs. (4.2) can be written as ($\eta = k_1 \tanh k_1(1 - k_1 \tanh k_1)$)

$$2\eta/\lambda = p - 2q. \tag{4.9}$$

Thus we have the system (4.1), (4.8), and (4.9) for determination of θ_n, λ_n, and G_n. From (4.1) and (4.7) we obtain

$$\hat{k}_2 = n\pi - \operatorname{arctg}(\sqrt{(m_1 + p)/(m_2 - p)}\ \operatorname{th} k_1); \tag{4.10}$$

$$\lambda = (m_2 - p)^{-1}(n\pi - \operatorname{arctg}(\sqrt{(m_1 + p)/(m_2 - p)}\ \operatorname{th} k_1))^2, \tag{4.11}$$

where n = 1, 2,... .

In constructing asymptotics, we will assume that p, q, m_1, $m_2 = O(1)$; λ, $\alpha^2 = O(n^2)$; $G = O(n^4)$; k_1, $\hat{k}_2 = O(n)$ for $n \to \infty$. This assumption is validated a posteriori: the asymptotic expansion is substantiated with the help of one of the singular variants of the implicit function theorem.

By replacing $\tanh k_1$ by 1, Eqs. (4.11) and (4.9) can be rewritten, with exponentially small error for $n \to \infty$, in the form

$$\lambda = (m_2 - p)^{-1}(n\pi - \operatorname{arctg}\sqrt{(m_1 + p)/(m_2 - p)})^2; \tag{4.12}$$

$$2(m_1 + p) + p - 2q = 2\sqrt{(m_1 + p)/\lambda}. \tag{4.13}$$

Bearing in mind that m_1, m_2, and p are expressed in terms of q by (4.6) and (4.8), one can treat (4.12), (4.13) as a system of equations which determine q and λ for a given n = 1, 2,.... After λ_n, q_n are found, the values of G_n, $\theta_n = \alpha_n^2$ are determined by the use of (4.4).

For limiting values of m_1, m_2, p, q determined from (4.13) as $n \to \infty$, the equation can be written as

$$(m_1 - 1)(2m_1 + 1)(2m_1 + 3) = 0. \tag{4.14}$$

As a result we find $m_1 = 1$, $m_2 = 2$, $p = 2/3$, and $q = 2$. From (4.4) and (4.12) we deduce the asymptotic equalities

$$\lambda_n = \frac{3}{4} n^2\pi^2, \quad \alpha_n = \frac{\pi n}{\sqrt{2}}, \quad G_n = \frac{27}{16} n^4\pi^4. \tag{4.15}$$

One can obtain asymptotic expansions for m_1, m_2, p, q in powers of 1/n:

$$m_1 = 1 + \sum_{k=1}^{\infty} m_{1k} n^{-k}, \quad m_2 = m_1 + 1, \tag{4.16}$$

$$p = \frac{2}{3} + \sum_{k=1}^{\infty} p_k n^{-k}, \quad q = 2 + \sum_{k=1}^{\infty} q_k n^{-k};$$

$$p_1 = 4\sqrt{5}/45\pi,\ q_1 = -4\sqrt{5}/5\pi, \quad m_{11} = -4\sqrt{5}/15\pi, \tag{4.17}$$

$$p_2 = \frac{248}{2025\pi^2} + \frac{4\sqrt{5}}{45\pi^2} \operatorname{arctg} \frac{\sqrt{5}}{2},$$

$$q_2 = -9p_2 + 32/(15\pi^2),\ m_{12} = -3p_2 + 16/(27\pi^2).$$

The numerical values are: $q_1 = -0.56941$, $p_1 = 0.0632677$, $m_{11} = 0.1898033$, $p_2 = 0.0293466$, $q_2 = -0.0479676$, and $m_{12} = -0.0279977$.

Using (4.12) we find, to an accuracy of O(1/n) for $n \to \infty$

$$\sqrt{\lambda_n} = \frac{\sqrt{3}}{2}\left(n\pi + \frac{2\sqrt{5}}{15} - \operatorname{arctg}\frac{\sqrt{5}}{2}\right). \tag{4.18}$$

Formula (4.18) gives reasonable results even for n = 1. The values of λ_n computed from this formula are: $\lambda_1 = 5.06(8)$, $\lambda_2 = 24.713(707)$, $\lambda_3 = 59.1655(2)$, $\lambda_4 = 108.4224(5)$, and $\lambda_5 = 172.484(8)$. Here the numbers in parentheses indicate the correct final digits (Table 1).

Odd Modes. The analysis is done in a completely analogous fashion for the odd modes. In place of (4.1), we have Eq. (2.4), and in (4.9) we take for η the expression $\eta = k_1 \coth k_1 (1 - k_1 \coth k_1)$. Correspondingly in (4.10), (4.11) n must be replaced by n + 1/2. The

TABLE 1

n	G	α	λ
1	729,1942188	1,9968426	5,0867521
2	23526,77881	4,1811270	24,706872
3	146200,8065	6,3965159	59,165166
4	509650,8013	8,6151200	108,42472
5	1317901,713	10,834913	172,48753
6	2838099,643	13,055278	251,35421
7	5400511,782	15,275961	345,02502
8	9398526,413	17,496839	453,50006
9	15288652,91	19,717846	576,77940
10	23590521,74	21,938942	714,86307

TABLE 2

n	G	α	λ
1	373,8634460	3,0760953	13,020459
2	4065,791645	5,2881467	40,086069
3	17909,79080	7,5056013	81,944607
4	52684,93200	9,7249200	138,60566
5	123115,3126	11,945044	210,07037
6	247870,0981	14,165589	296,33909

limiting values of the parameters m_1, m_2, p, q for $n \to \infty$, asymptotic formulas (4.15), and expressions (4.17) are preserved. In place of (4.18) we obtain

$$\sqrt{\lambda_n} = \frac{\sqrt{3}}{2}\left(n\pi + \frac{\pi}{2} + \frac{2\sqrt{5}}{15} - \operatorname{arctg}\frac{\sqrt{5}}{2}\right). \tag{4.19}$$

The error here is as before: $O(1/n)$ for $n \to \infty$. Of course, this formula works even better than that for the even modes. For example, according to (4.19) we have $\lambda_1 = 13.04$ and $\lambda_2 = 40.089$. The computed values (Table 2) are $\lambda_1 = 13.020459$ and $\lambda_2 = 40.086069$.

5. Numerical Results. We write Eq. (4.9) in the form

$$m_1^4 + m_1^3 + (B - 1{,}25)m_1^2 + (B - 0{,}75)m_1 - B/2 = 0, \tag{5.1}$$

where

$$B = \sqrt{(m_1 + p)/\lambda}\ \mathrm{th}\, k_1 + (m_1 + p)(1 - \mathrm{th}^2 k_1), \qquad k_1 = \sqrt{\lambda(m_1 + p)}, \quad p = q/(2q - 1), \quad q = m_1^2 + m_1. \tag{5.2}$$

By combining these with (4.11), we obtain a system of equations for the even modes. For the odd modes, we need only replace tanh by coth in the expressions for B and λ_n, and replace n by n + 0.5 in (4.11).

System (4.11), (5.1), and (5.2) is solved numerically in the following fashion. Having an approximate value for λ_n (at worst one can begin with $\lambda_n^0 = \infty$), we compute B and find m_1 by solving the fourth degree equation (5.1). By using Descartes' law, we establish that the last equation has a single positive root, which is determined by Newton's method. Finding m_1, we compute λ_n according to (4.11) and (5.2), and α_n and G_n by (4.4).

The numerical results are shown in Tables 1 and 2.

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